Student Teachers' Use of Technology-Generated Representations: Exemplars and Rationales

Nicole Juersivich
University of Virginia
nrm3w@virginia.edu

Joe Garofalo and Virginia Fraser
University of Virginia

Keywords: Teacher Education, Technology, Mathematics, Multiple Representations

Since the 1990s, technology has become increasingly available in classrooms. Because of this, the National Council of Teachers of Mathematics (NCTM) designates technology as one of the fundamental principles in "high-quality mathematics education" (2000, p.11), for "it influences the mathematics that is taught and enhances students' learning" (p. 24). NCTM argues that the use of technology can promote deeper understanding, since it can empower students to discover, explore, and conjecture about mathematical ideas; thus, it allows students to act and think as mathematicians by fostering their conceptual understanding (Borwein & Bailey, 2003; Chazan, 1999; Heid, 1997).

Multiple Representations

One way that teachers can use technology to facilitate their students' mathematical understanding is through generating and exploring multiple representations of mathematical ideas. Three prominent external representations are used in mathematics: graphs, tables, and equations (Ainsworth, 1999; Brenner et al, 1997; Goldin, 2002; NCTM, 2000). Each of these has special characteristics that emphasize certain aspects of a concept. Graphical representations show a consolidated, larger, and continuous view of a concept's structure, whereas tabular representations present an explicit, smaller, discrete view of information (Kaput, 1989, p. 172). Algebraic representations express the succinct, symbolic relationship between variables to demonstrate how a change in one variable will affect a change in another. Because of the advantages of each representation, they are important for students to learn in their own right. The overall benefit is that the knowledge of one representation can assist in interpreting another, and hence “promote deeper understanding of the domain” (Ainsworth, Bibby, & Wood, 1997, p. 94), by enhancing one’s concept images, which are comprised of all the possible figures, ideas, and surrounding thoughts that reference particular concepts (Vinner & Dreyfus, 1989).

Classroom Technology and Mathematical Representations

NCTM encourages the use of technology as a delivery medium of multiple representations because it “affords access to visual models that are powerful but that many students are unable or unwilling to generate independently" (p. 25) and can highlight the best qualities of each representation to make it easier for students to draw connections among them (Zbiek, Heid, & Blume, 2007). These capabilities are particularly valuable because many students have difficulty linking and moving among representations (Goldenberg, 1988; Leinhardt, Zaslavsky, & Stein, 1990). Furthermore, Jiang and McClintock (2000) petition mathematics educators to “make the best use of multiple representations, especially those enhanced by the use of technology" (p.19).

While technology can improve the quality of mathematical instruction, it does not do so by itself. To reap the benefits of technology, one must incorporate appropriate content with appropriate content pedagogy. Researchers agree that mathematics teachers are the vital element in the movement towards proper incorporation of technology into the mathematics classroom (Bottino & Furinghetti, 1996; Kaput, 1992; NCTM 1991, 2000). However, educating teachers in how to use the technology to extricate student-learning benefits is difficult and requires continuous and rigorous professional development and support (Mergendoller, 1994; Waits & Demana, 2000). In fact, early studies have found that it takes between three to five years before teachers become competent and feel confident enough to teach with technology (Dwyer, Ringstaff, & Sandholtz, 1991; Means & Olson, 1994).

As teacher educators, we have limited time and opportunity to prepare pre-service mathematics teachers (PSMTs) to use technology to foster their pupils' conceptual understanding of mathematics through the use of multiple representations. Fortunately, since present-day pre-service teachers are products of a technological culture, they are coming into teacher education programs with more technology experiences than their counterparts in the early 90s; hence, we believe it is possible to effectively prepare them to use technology appropriately in two years (Garofalo, Shockey, Harper, & Drier, 2000; Harper, Schirack, Stohl, & Garofalo, 2001).

Purposes

The purposes of this study are to: (1) observe and describe how PSMTs, who had a variety of experiences doing and practice-teaching mathematics in a technology-rich environment, subsequently incorporated technology-generated representations into their student teaching and (2) understand and expound on their rationales for using such representations. This study is part of a larger study of secondary science and mathematics pre-services teachers' use of visualization in whole class inquiry.

Methodology

Participants

Two cohorts of University of Virginia PSMTs participated in this study. The first cohort consisted of 10 PSMTs who student-taught in Fall 2005. The second cohort consisted of 6 PSMTs who student-taught in Fall 2006.

Secondary Mathematics Teacher Preparation

Secondary mathematics pedagogy course. Prior to student teaching, in the fourth year of a five-year program, all PSMTs spend two semesters in a technology-rich mathematics pedagogy course. This course focuses on problem-solving in the spirit of Pólya (1945) and Schoenfeld

(1985), student understanding and sense-making as discussed in Hiebert and LeFevre (1986), constructivism, and the NCTM (2000) *Principles and Standards*. The course integrates a variety of technologies in the context of meaningful mathematical activities and emphasizes using technology to generate multiple representations (Garofalo et al, 2000). The technologies that PSMTs use are graphing calculators and TI SmartView, Geometer's Sketchpad (GSP) and related products, Excel, CBLs with probeware, Green Globs, Explore Learning, and Flash applets. The PSMTs use these technologies to explore mathematical problems and concepts in class and for homework and subsequently reflect on these experiences. Through self-reflection, they often realize that they lack conceptual knowledge about certain topics, which they later develop while working with the technology.

PSMTs not only use these technologies to do mathematics, but they learn different ways to incorporate technology into their teaching. They use these technologies in conjunction with a SMART Board and SMART Notebook in mini-lessons that they teach in the course and use technology in the field experiences in local high schools (sometimes without a SMART Board). We provide feedback on pedagogy and technology use to the PSMTs following each lesson. After doing and teaching mathematics with technology, PSMTs often make comments such as 'I wish I would have learned it this way.'

Prior to entering this pedagogy course, all PSMTs have basic familiarity with Excel, PowerPoint, and graphing calculators. Some of these technology experiences are gained during an introductory course on educational technology. Some PSMTs have knowledge of GSP; however, thus far, none had used TI SmartView or a SMART Board.

Student teaching. Researchers have recognized that simply using technology to learn mathematics will not guarantee that teachers will use it in their teaching, for many factors confound the initiative to use it (Olive & Leatham, 2000). PSMTs need access to equipment along with pedagogical and technical support. To enable our PSMTs to implement the technologies they learned, we provide them with a laptop, a SMART Board, a projector, and software to use during their student teaching in their fifth year. PSMTs are observed and debriefed regularly and meet weekly as a group to discuss teaching and technology issues. Furthermore, during these seminars, PSMTs share their technology files and talk about how they used them in their classrooms.

Given that PSMTs had time to use and understand how technology can be incorporated into the mathematics classroom, the researchers wanted to know how these PSMTs implemented technologies in their student teaching placements.

Data Collection

Consistent with the purposes of this study, the data collection is both descriptive and qualitative. Data were collected through means such as *observations, interviews,* and *artifacts.*

Observations and debriefings. During student teaching, we observed at least five teaching episodes per PSMT. Teaching observations lasted 90 minutes with observers focusing on the following: student engagement, meaningful learning, technology use, lesson context, behavior/classroom management, rapport, and mathematical aspects. Observation notes were taken with respect to these factors. Following the observation, the observer and PSMT held a half-hour debriefing session in which the PSMT commented on what worked well within the classroom, what needed more work and how to go about improving them, the PSMT's planning process, and ideas about how and when to use technology. These debriefings were audio recorded and transcribed.

Interviews. Prior to teaching in their placements, we interviewed the PSMTs concerning their views on planning, effective mathematics lessons, technology benefits to teaching and learning, and the primary motive for using technology. After student teaching, we asked similar questions and had the PSMTs provide examples from their placements. These interviews were audio recorded, transcribed, and imported into the computer qualitative data analysis software *NVivo7*.

Reflective journal. During their student teaching placements, the PSMTs kept an electronic technology reflective journal. In this journal they were asked to write weekly about any feelings, thoughts, or issues they had with the technology in their field placement and post it to a secure site. These documents were imported into *NVivo7* as well.

Teaching artifacts. During their student teaching, PSMTs saved sample technology files and lesson plans on a common share drive so that everyone could access them. Later in the semester during the seminar, PSMTs demonstrated their best lessons and technology files and gave a synopsis of how they incorporated technology into their lesson.

Data Analysis

We adopted Erickson's analytic induction (1986) to analyze the data, where the first phase of data analysis is to generate assertions by carefully searching through the body of data. The second phase, to establish evidentiary warrant for each assertion, is done by performing a "systematic search of the entire data corpus, looking for disconfirming and confirming evidence, keeping in mind the need to reframe the assertions as the analysis proceeds" (p. 146).

We imported the transcribed reflective journals and interviews into *NVivo7*. Following data entry, we went line-by-line through each document and open-coded discrete chunks of information by using descriptive, conceptual, and in vivo codes. After open-coding, the researchers noticed the emergence of two main codes: *Quality of Life* and *Representation*; this paper solely focuses on the latter. The researchers then began to make relationships among the remainder of the codes and *Representation* in a hierarchical fashion. During this process, codes were refined by defining and renaming through the use of memo-ing and writing about the meanings of codes.

Under *Representation*, three main themes surfaced across and among participants. These themes were exposed through readings and re-readings of the information within node and subnodes of *Representation*. These themes were revised and eventually became the three assertions. These assertions were authenticated by systematically searching for warrant and assisted by analytic memo-ing.

Results

The first assertion characterizes how and why PSMTs used technology-generated representations in their instruction. PSMTs believed that technology allowed their pupils to gain a deeper understanding of certain mathematical concepts through their active involvement. The last two assertions also deal with pupil understanding, but less directly. Assertion 2 points out how the use of quickly generated representations facilitated understanding by removing the time consuming tasks of mechanically constructing them. By eliminating time-arduous tasks, PSMTs were able to spend more time asking questions, and pupils were able to spend more time interpreting and reasoning from the multiple representations. Assertion 3 shows how technology-generated representations could enable understanding because it eliminated possible misconceptions brought about by inaccuracies and lack of precision of figures and graphs.

Assertion 1: PSMTs frequently used technology to generate representations, not otherwise feasible, for the purpose of facilitating their pupils' conceptual understanding. In particular, they incorporated dynamic representations to promote pupils' discovery and generalization through active prediction, manipulation, observation, and interpretation.

While the majority of the PSMTs used technology-generated representations almost everyday, the rest did at least weekly. These included both pre-existing and PSMT-created representations. PSMTs believed that dynamic representations, such as adjustable figures, charts, and graphs, allowed their pupils to see underlying mathematical structure. Since pupils manipulated and observed representations, they were able to visualize intrinsic properties. They controlled the movement of objects and received instantaneous feedback, from the technology or the teacher, and hence generalized about properties of concepts. PSMTs encouraged participation through active prediction and interpretation of ideas, so that pupils took part in their learning.

Example 1.1: A pre-existing derivative file. D11 used the GSP file shown in Figure 1 to dynamically show how the secant line becomes the tangent line by letting *h* approach zero in the following equation: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. GSP dynamically illustrated the limiting process and allowed pupils to see the connection between the graphic and algebraic representations of slope. By observing and reflecting on this motion, the pupils could better conceptualize this process and understand what they were "calculating" when they referred back to the algebraic notation.

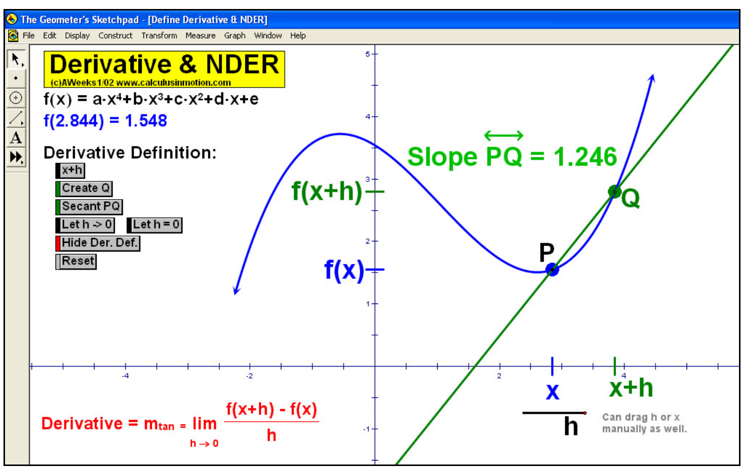

Figure 1. Derivative GSP

In her reflective journal, D11 described the effect of using GSP to illustrate derivatives.

> This [GSP] was a huge help in explaining derivatives (rate of change). It allowed students to see (in motion) the changing of a tangent line with respect to the change of the value of *h* in the definition of the limit. This extremely aided the theoretical understanding of what the limit is and how it translates to applicable and concrete understanding (and subsequently calculating derivatives in actual problems).

Example 1.2: A pre-existing integration applet. D2 imported our in-house developed Integration Tool (http://www.teacherlink.org/content/math/interactive/flash/home.html) into her SMART Notebook and executed it by tapping on the SMART Board.

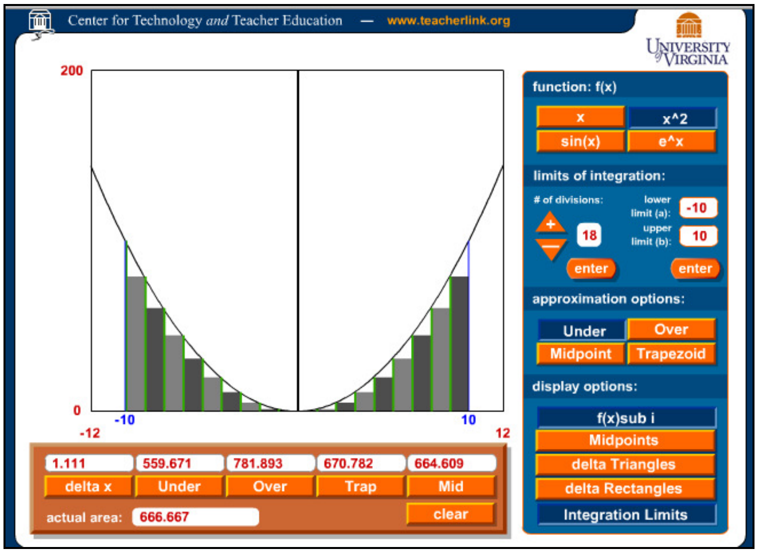

Figure 2. Integration Tool

This tool gave pupils the opportunity to change the function, limits of integration, and approximation techniques (see Figure 2). Parameters, such as the number of subdivisions, were changed easily, and pupils observed the outcomes of these changes. This enabled them to connect integration with the limiting process of successive approximation. These dynamic aspects of the tool proffered pupils a greater understanding of integration. D2 discussed how useful this flash tool on a SMART Board was rather than transparencies on the overheard when teaching the concept of integration.

> I just think that the possibilities for student understanding are just so much greater. If you can show a dynamic thing of integration versus just a couple stationary pictures, students are going to get something that seems more real and visual. (Exit Interview)

Example 1.3: A PSMT-created matrix file. As one PSMT was beginning the chapter on matrices, she realized that she needed some kind of visualization to emphasize the procedure of matrix multiplication, because her pupils seemed to have a difficult time remembering which column and row to multiply.

> I would need some sort of visual that would explain how matrix multiplication works… just saying it wasn't clicking…they needed to know what these numbers mean. Movement can highlight what numbers are used in each step. (A2, Exit Interview)

She wanted to make matrix multiplication relevant to her pupils, so she situated the idea in the context of buying pizza, breadsticks, and soda from three popular pizza places for two different algebra classes. Her matrices consisted of a price matrix multiplied by a quantity matrix in order to get the total cost of the items for each class. She used the animation features in PowerPoint to highlight the appropriate row and column and then trace the path to where the numbers would enter into the new matrix.

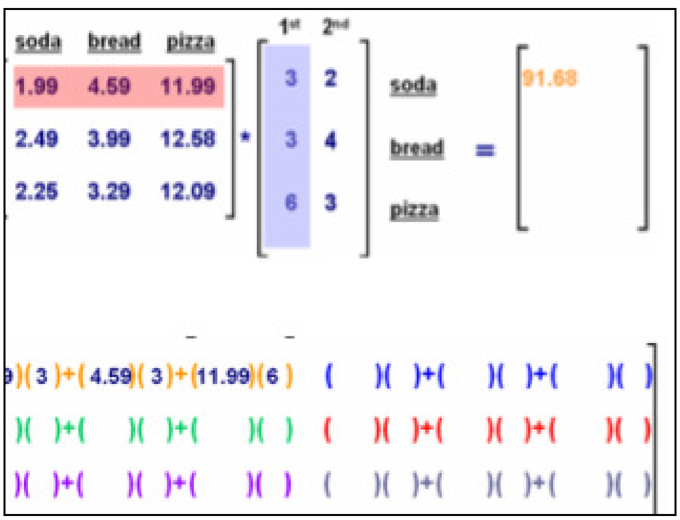

Figure 3. PowerPoint Matrix Multiplication

She noted that by having pupils see each pair of numbers move to the correct position in the new matrix, rather than a static picture from a book, her pupils could understand the rationale behind the procedure of matrix multiplication.

Example 1.4: A PSMT-created absolute value function file. PSMTs used technology to allow pupils to manipulate to discover and generalize about the structure of absolute value functions.

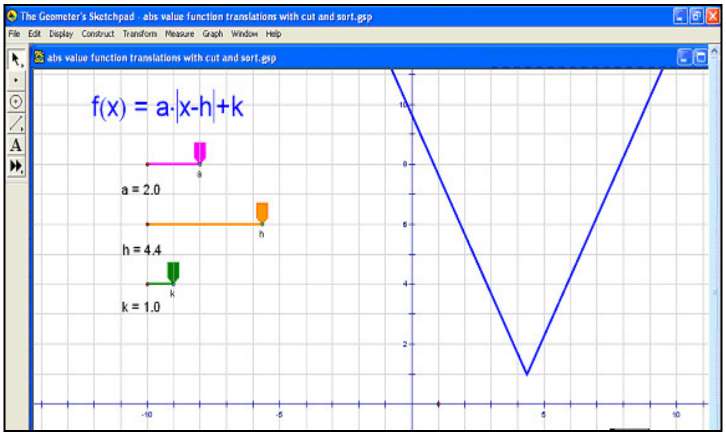

Figure 4. Absolute Value Transformations

Pupils were able to move the sliders in the GSP file (see Figure 4) and observe how the changing of different parameter values in the equation, $f(x) = a|x\text{-}h| + k$, immediately affected the graph.

> There was a slider attached to each parameter *h* and *k*…so students could click on *h* and move it around and see the graph move from left to right and right to left. And they could say, 'Oh, that's a horizontal transformation.' It was kind of guided manipulation…to help them…while they figured it out. (L2, Exit Interview)

Summary

PSMTs regularly used technology-generated representations in mathematics topics ranging from algebra to calculus because they believed that technology aided in their pupils' conceptual and procedural understanding. While most PSMTs who taught calculus relied on ready-made representations, other PSMTs who taught algebra felt the need to create representations in GSP and PowerPoint to aid their pupils' understanding of certain topics.

Most of the issues that PSMTs discussed in calculus centered on pupils not being able to understand derivatives and integrals by their algebraic definitions. All calculus PSMTs felt that the incorporation of dynamic representations allowed their pupils to see the mathematics more clearly in order to understand and construct the appropriate algebraic equation. Visualizing the idea behind the algebraic notation is challenging for pupils even if the teacher can draw the images. Through the use of technology, pupils witnessed the limiting process in both the definition of derivative and integration so they could focus on finding the derivative or integrating on the algebraic representations versus constructing both the graphical and algebraic representations at the same time in order to fully understand what was occurring.

PSMTs who taught algebra felt that technology aided in pupils' conceptual understanding. PSMTs noticed a difference in teaching a concept when the pupils were able to manipulate objects. The pupils were able to discover innate structures by conjecturing and then testing their conjectures. The technology provided them with objects on which to operate as well as immediate feedback. Once pupils received the feedback, they altered their conjecture and tested their new hypothesis.

PSMTs asserted that there was an initial learning curve involved in understanding the features of the provided technologies and how to use them to generate effective representations. They learned how to use many of the basic and intermediate features during the mathematics pedagogy course, but sometimes they overcame other challenges during student teaching when learning to use some more advanced features.

Assertion 2: By using technology-generated representations, PSMTs were able to save class time normally needed for manual graphing and drawing, thus providing more time for analyzing representations.

Many PSMTs complained about how long it took to draw on the board. They felt the time it took to draw could be used instead for more in-depth learning through the questioning of and analysis of representations. In her exit interview, C2 mentioned how technology benefited her because without it, "it takes time … to write your function, to draw your graph, to make it look good, put it to scale, label your points. It takes a lot of time to do that." Like other PSMTs, she created her representations using GSP or TI SmartView and inserted them into her SMART

Notebook, which was then projected for the pupils to see. She felt that the use of technology freed her from the time it took to draw difficult diagrams on the board.

By using the interactive, projected calculator, TI SmartView, and displaying it on the SMART Board, pupils and PSMT could graph multiple functions more quickly than if they had done so by hand. This allowed pupils to conjecture how the algebraic form of the equation impacts the graph of the equation. If the pupils and PSMT had to draw all these graphs by hand, it would take longer to come to a generalization.

> This week we focused on transformations; here the graphing calculator facilitated much quicker and more in depth learning. Students could quickly graph functions and hypothesize how changing the equation would change the graph, or vice versa. (T1, Reflective Journal)

T1 liked the fact that by using TI SmartView on a SMART Board, PSMTs could tap on the board to make quick alterations to the equation, which would immediately alter the corresponding table and graph. Since SmartView elicits a *Cube View* (see Figure 5) in which it shows all three representations at one time, pupils can generalize and discover properties of transforming functions through the immediate changes.

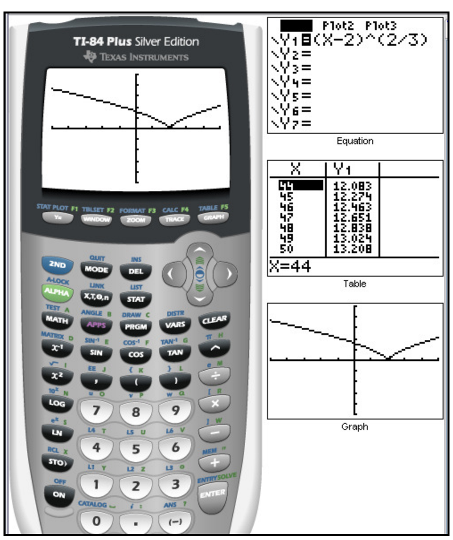

Figure 5. Cube View in TI SmartView

While the technology did not supply the pupils with the correct conjecture or generalization, it did facilitate the mechanical processes of drawing so that pupils spent more time analyzing relationships among representations. M2 noticed this in his classroom:

> The technology doesn't give the meaning, but it makes the process quicker and then you can get at the meaning… I was able to ask a lot of questions about what does it mean, predicting things like that…allow more time for understanding and getting at the deeper level. (Exit Interview)

Summary

Pupils' understanding of a concept will not improve by simply supplying multiple representations, for they need to know what information these representations are bringing forth as well as how to connect these representations (Ainsworth et al, 1997; Meyer, Shinar, & Leiser, 1997). Furthermore, the fact that technology was used to generate the representations and displayed all three types did not provide meaning to the pupils; however, it did allow teachers and pupils to create the representations more quickly than they normally could. The extra time provided by the technology was used by teachers to question and involve their pupils in thinking mathematically by analyzing the representations.

Assertion 3: PSMTs claimed that the accuracy of the figures and graphs generated with technology facilitated pupils' conjecturing, analyzing, and deducing.

PSMTs noticed that when they manually drew a graph to emphasize part of a concept, sometimes their effort would be fruitless because of the lack of accuracy and precision that was required to fully grasp the importance of using the graphical representation.

Example 3.1 Generation of correct graph. T1 became aware of his inability to depict a truthful graph of a function, so he utilized the TI SmartView software to create a precise graph. In his reflective journal, T1 noted that his lesson focus was on asymptotes. During his class, he wanted to use the function $f(x)=\frac{1}{x}$ (Figure 6a) to show the algebraic and graphical representation of an asymptote, yet he inadvertently graphed $-f(x)$ (Figure 6b) instead. Because he was so consumed with making his point about asymptotes, he sacrificed the correct function, which may have led pupils to confusion, for the graph and the equation were not equivalent.

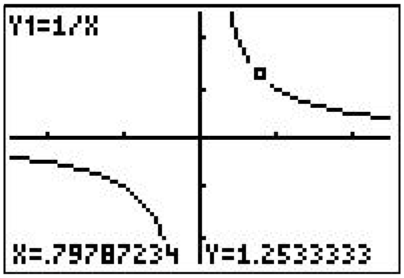

Figure 6a. $f(x)=\frac{1}{x}$

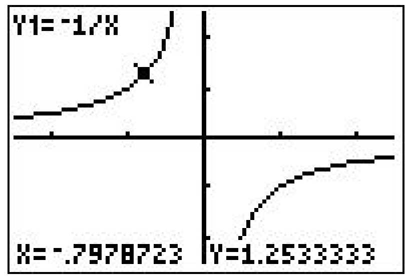

Figure 6b. $f(x)=-\frac{1}{x}$

> When it comes to drawing graphs, my own handwritten graphs are sometimes difficult to interpret. When hand-drawing graphs like $\frac{1}{x}$, I may unintentionally display the graph as increasing as x approaches infinity, instead of decreasing. Though this is usually done in an effort to ensure the graph maintains its asymptote of $y = 0$, the graphing calculator is much more accurate, and through SmartView, the students can witness this accuracy while confusion is minimized. [Reflective Journal]

Example 3.2 Generation of precise measurements. Since GSP provides precise measurements of figurate angles and sides, pupils can visualize properties of figures and the natural consequences that might precede a formal proof. This visualization leads to better conjectures and inductive thinking. Furthermore, using GSP allows pupils to discover generalizations by themselves, which changes the teacher's role from a director to a facilitator. K1 stated this phenomenon in his reflective journal.

> It [GSP] is extremely useful in helping visualize concepts, and adds precision and ease to the lesson. It also facilitates discovery learning and inductive lessons through the access to GSP and other tools. Observing and making conjectures becomes easier.

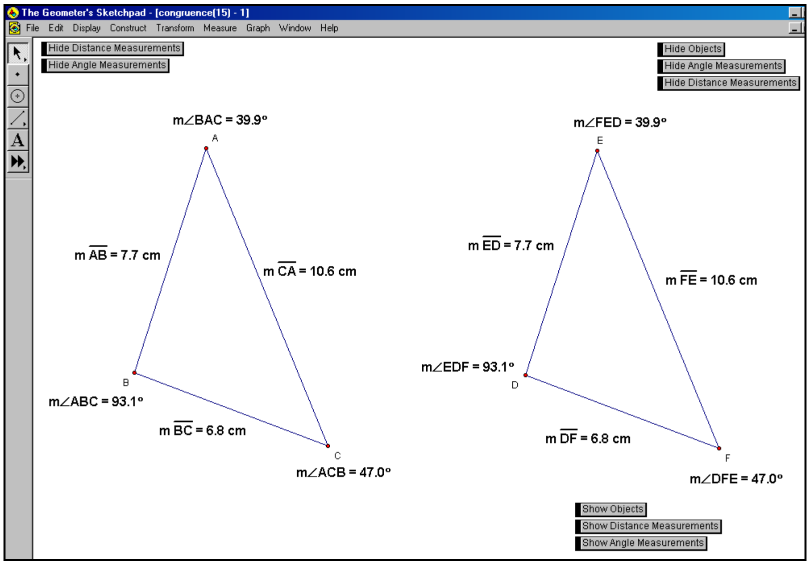

Figure 7. GSP Measurement of Triangles

Figure 7 is a GSP file created by K1 that he used in a lesson involving congruent triangles. He constructed two triangles and labeled the side lengths and angle degrees by using GSP measurement features. Pupils were able to manipulate the triangles and to notice the numerical similarities and differences that occur when triangles are congruent. Through manipulation and observation, pupils were able to discover the necessary and sufficient properties for triangle congruency.

Summary

Using TI SmartView and GSP, pupils were able to reason from accurately portrayed diagrams. Because the diagrams were perfectly illustrated, pupils spent time making appropriate generalizations and connections among representations, without becoming confused due to the lack of correctness. Prior to technology, pupils had to rely on hand-drawn, potentially inaccurate diagrams from which to reason. Most likely, this is the reason for George Pólya's statement: "Geometry is the science of correct reasoning on incorrect figures" (1945). In fact, one PSMT, M2, relayed exactly that finding in his exit interview.

> [D]rawing triangles, you tell the students, 'Oh be exact.' But then you half-ass draw a triangle and you're, 'Look an isosceles triangle.' And your kids are, 'That doesn't look anything like an isosceles triangle.'

With the incorporation of technology in the mathematics classroom, pupils and PSMTs alike did not need to have poorly drawn diagrams on which they had to base their proof.

Discussion

We found that these PSMTs, after participating in the mathematics pedagogy course described above, appropriately utilized technology regularly during their student teaching to generate representations. They did so for the purpose of helping their pupils develop conceptual and procedural understanding of mathematics.

These PSMTs firmly believed the dynamic visualization afforded by technology supports pupils' sense-making in ways that could not be actualized under typical conditions. PSMTs felt that because technology-generated representations are easily manipulated, pupils can visualize the relationships that are being represented and observe the consequences of their actions. PSMTs professed that technology expedited the generation of graphs, tables, and equations that would normally monopolize instructional time, so teachers and pupils were free to focus on analyzing and discussing the connection among and the meaning behind the representations; thereby, raising learning expectations. PSMTs used the technology to produce accurate, manipulatable representations, which pupils explored, conjectured on, and analyzed. Through guided discovery, pupils could appreciate the best features of each representation as well as the connection among the representations.

We feel that the ongoing experiences of doing mathematics with technology during their preparation program enabled PSMTs to see the potential of technology to permit instruction otherwise not feasible and to develop a vision for their own technology uses. They had multiple opportunities to explore mathematics concepts and applications with different technologies, and practice teaching with these technologies. The PSMTs came to believe that multiple and dynamic representations enhanced their own understanding of mathematics, and this persuaded them to incorporate technology during their student teaching.

These PSMTs did not previously view the use of technology as a way to foster their pupils' conceptual understanding. Prior to the pedagogy course, these PSMTs professed their apprehensions about the use of technology. They expressed either their lack of knowledge of how to incorporate technology or their concern about the detrimental effects of classroom technology use. For example, E2 stated during the first period of the course, "Technology use in mathematics is somewhat new to me…I still don't feel that I know much about using it in the classroom." Upon completion of the course, E2 asserted, "Effective math teaching *has* to incorporate some of the new math technologies that have become available. I think it makes mathematics much more useful and 'modern' from the perspective of the students. Also, there is a lot that can be done with technology that can't be done otherwise."

Those PSMTs who were initially cautious about using technology believed that pupils should be able to do mathematics without it. M2 remarked, "I believe students should be able to do problems without it, so that they can understand the process of how the technology derives the answer." L2 similarly commented, "I think it's overused and overemphasized at basic levels. Students should be able to solve problems on their own not just know their way around a computer screen. Technology should be supplemental and not central." Thus, they believed technology functioned mainly as a computational device rather than a learning resource. Yet following the course, their views grew to include ideas about using technology to promote pupil learning. M2 later proclaimed, "Utilizing interactive technology to make learning more hands on that paper, pencil, ruler etc. I plan on using as much as possible." L2 testified, "Technology helps fulfill the goals of the differentiated classroom. Some students learn by doing or visually and technology enables that to happen." We attribute the changes in disposition to the multitude of opportunities these PSMTs had that allowed them to experience the benefits of technology by learning or *relearning* some mathematical topics through technology and seeing new possibilities for teaching with technology-generated representations.

Clearly, this study has some limitations. We studied our own students, and it is possible that some of their actions and words were tailored to conciliate us. In addition, this study examined a small participant group in a unique setting where both technology and support was available in every student teaching classroom. As this study is bound in context, it does not readily generalize to all PSMTs without further research. However, the program description and results reported here may give teacher educators some ideas to use in their own preparation programs.

References

Ainsworth, S. (1999). The functions of multiple representations. *Computers & Education*, 33, 131-152.

Ainsworth, S., Bibby, P. & Wood, D. (1997). Information technology and multiple representations: new opportunities – new problems. *Journal of Information Technology for Teacher Education*, 6(1), 93-105.

Borwein, J. M., & Bailey, D.H. (2003). *Mathematics by experiment: Plausible reasoning in the 21st Century.* Natick, MA: AK Peters.

Bottino, R., & Furinghetti, F. (1996). Emerging of teachers' conceptions of new subjects inserted in mathematics programs: The case of informatics. *Educational Studies in Mathematics*, 30, 109-134.

Brenner, M., Mayer, R., Mosely, B., Brar, T, Durán, R., Reed, B. & Webb, D. (1997). Learning by understanding: The role of multiple representations in the learning of algebra. American Educational Research Journal, 34(4), 663-689.

Chazan, D. (1999). On teachers' mathematical knowledge and student exploration: A personal story about teaching a technologically supported approach in school algebra. *International Journal for Computers in Mathematical Learning*, 4, 121-149.

Dwyer, D.C., Ringstaff, C., & Sandholtz, J.H. (1991). Changes in teachers' beliefs and practices in technology-rich classrooms. *Educational Leadership, 48* (8), 45-52.

Erickson, Frederick. (1986). Qualitative Methods in Research on Teaching. In M.C. Wittrock (Ed.) *Handbook of Research on Teaching* (pp. 68-120). New York: Macmillan.

Garofalo, J., Shockey, T., Harper, S., & Drier, H. (2000). The Impact project at Virginia: Promoting appropriate uses of technology in mathematics teaching. *Virginia Mathematics Teacher*, *25* (2), 14-15.

Goldenberg, E.P. (1988). Mathematics, metaphors, and human factors: Mathematical, technical, and pedagogical challenges in the educational use of graphical representations. *Journal of Mathematical Behavior*, *7*, 135-173.

Goldin, G. (2002). Representation in mathematical learning and problem solving. In L. English (Ed.), *Handbook of international research in mathematics education* (pp. 197-218). Mahwah, NJ: Lawrence Erlbaum Associates.

Harper, S.R., Schirack, S.O., Stohl, H.D., & Garofalo, J. (2001). Learning mathematics and developing pedagogy with technology: A reply to Browning and Klepsis. *Contemporary Issues in Technology and Teacher Education,* [Online serial] *, 1* (3) . Available: http://www.citejournal.org/vol1/iss3/currentissues/mathematics/article1.htm

Heid, M.K. (1997). The technological revolution and the reform of school mathematics. *American Journal of Education*, 106(1), 5-61.

Hiebert, J. & LeFevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.) *Conceptual and procedural knowledge: The case of mathematics* (pp. 1-27). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Jiang, Z., & McClintock, E. (2000). Multiple approaches to problem solving and the use of technology. *Journal of Computers in Mathematics and Science Teaching, 19* (1), 7-20.

Kaput, J.J. (1989) Linking representations in the symbolic systems of algebra. In S. Wagner & C. Kieran (Eds.) *Research issues in the learning and teaching of algebra (*pp 167-194). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Kaput, J.J. (1992). Technology and mathematics education. In D.A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning (pp. 515-556)*. New York: Macmillan.

Leinhardt, G., Zaslavsky, O., & Stein, M.K. (1990). Functions, graphs, and graphing: Tasks, learning and teaching. *Review of Educational Research*, *60* (1), 1-64.

Means, B., & Olson, K. (1994). Tomorrow's schools: Technology and reform in partnership. In B. Means (Ed.), *Technology and the education reform* (pp. 191-222). San Francisco: Jossey-Bass.

Mergendoller, J.R. (1994). The Curry School of Education, University of Virginia. In *Exemplary approaches to training teachers to use technology, vol. 1: Case studies* (pp. 4.1-4.24). Novato, CA: Beryl Buck Institute for Education.

Meyer, J., Shinar, D., & Leiser, D. (1997). Multiple factors that determine performance with tables and graphs. *Human Factors*, 39(2), 268-286.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics* . Reston, VA: Author.

National Council of Teachers of Mathematics. (2000) *Principles and standards for school mathematics.* Reston, VA: NCTM.

Olive, J., & Leatham, K. (2000). *Using technology as a learning tool is not enough* . Paper presented at the International Conference of Technology in Mathematics Education, Auckland, New Zealand.

Pólya, G. (1945) *How to Solve it?* Princeton, NJ: Princeton University Press.

Schoenfeld, A. (1985). *Mathematical problem solving.* New York: Academic Press.

Vinner, S. & Dreyfus, T. (1989) Images and definitions for the concept of function. *Journal for*

Research in Mathematics Education, 20(4), 356-366.

Waits, B.K., & Demana, F. (2000). Calculators in mathematics teaching and learning: Past, present, and future. In M. J. Burke and F. R. Curcio (Eds.), *Learning mathematics for a new century* (pp. 51-66). Reston, VA: National Council of Teachers of Mathematics.

Zbiek, R M.., Heid, M. K., Blume, G., & Dick, T.P. (2007). Research on technology in mathematics education: The perspective of constructs. In F. K. Lester (Ed.), *Second handbook of research in mathematics teaching and learning* (pp. 1169-1207). Charlotte, NC: Information Age Publishing.

Note: The research reported here was funded by a grant from the U.S. Department of Education Fund for the Improvement for Postsecondary Education.